OLIMPIADA NAȚIONALĂ DE MATEMATICĂ
FAZA LOCALĂ 12.02.2024

BAREM DE CORECTARE ȘI NOTARE- CLASA a XI-a

SUBIECTUL I (7 puncte)

Fie $M(x,k) = \begin{vmatrix} \sqrt[3]{x} & \sqrt[3]{k} & \sqrt[3]{x}+\sqrt[3]{k} \\ \sqrt[3]{k} & \sqrt[3]{x}+\sqrt[3]{k} & \sqrt[3]{x} \\ \sqrt[3]{x}+\sqrt[3]{k} & \sqrt[3]{x} & \sqrt[3]{k} \end{vmatrix}$, **cu x, k** $\in \mathbb{R}$.

a) **Să se calculeze M (-k, k) si E** $(\sin^2\alpha, \cos^2\alpha)$, $\alpha \in \mathbb{R}$. **5p**

b) **Să se rezolve în mulțimea numerelor reale inecuația** $\sum_{k=1}^{4047} M(x,k) \geq 0$. **2p**

Aducem la forma cea mai simpla expresia data. Se aduna toate coloanele la prima coloana. Se scoate factor comun si se obtine :

$$M(x,k) = 2\cdot(\sqrt[3]{x}+\sqrt[3]{k}) \begin{vmatrix} 1 & \sqrt[3]{k} & \sqrt[3]{x}+\sqrt[3]{k} \\ 1 & \sqrt[3]{x}+\sqrt[3]{k} & \sqrt[3]{x} \\ 1 & \sqrt[3]{x} & \sqrt[3]{k} \end{vmatrix}.$$

Se scade din linia a doua prima linie si din a treia linie scadem a doua linie.

Obtinem : $M(x,k) = 2\cdot(\sqrt[3]{x}+\sqrt[3]{k})(\sqrt[3]{x}\cdot\sqrt[3]{k} - \sqrt[3]{x} - \sqrt[3]{k}) = -2(x+k)$; (1) ………..……………...3p

a) $M(-k,k) = \begin{vmatrix} -\sqrt[3]{k} & \sqrt[3]{k} & 0 \\ \sqrt[3]{k} & 0 & -\sqrt[3]{k} \\ 0 & -\sqrt[3]{k} & \sqrt[3]{k} \end{vmatrix} = \begin{vmatrix} 0 & \sqrt[3]{k} & 0 \\ 0 & 0 & -\sqrt[3]{k} \\ 0 & -\sqrt[3]{k} & \sqrt[3]{k} \end{vmatrix} = 0$ ………………………………1p

$M(\sin^2\alpha, \cos^2\alpha) = -2(\sin^2\alpha + \cos^2\alpha) = -2$ ………..…………………..………………………1p

b) Folosind (1), obtinem :
$-2(x+1) - 2(x+2) - \ldots - 2(x+4047) \geq 0.$
$4047x + (1+2+3+\ldots+4047) \leq 0 \Rightarrow x \leq -2024$ …………………………………………..2p

SUBIECTUL II (7 puncte)

Să se calculeze: $\lim\limits_{n\to\infty}(\sqrt{n^2+n}-\sqrt[3]{n^3+n^2})$

$\lim\limits_{n\to\infty}(\sqrt{n^2+n}-\sqrt[3]{n^3+n^2}) = \lim\limits_{n\to\infty}[(\sqrt{n^2+n}-n)-(\sqrt[3]{n^3+n^2}-n)] =$

$= \lim\limits_{n\to\infty}(\sqrt{n^2+n}-n) - \lim\limits_{n\to\infty}(\sqrt[3]{n^3+n^2}-n)$ ……………………………………………………2p

$\lim\limits_{n\to\infty}(\sqrt{n^2+n}-n) = \lim\limits_{n\to\infty}\frac{n^2+n-n^2}{\sqrt{n^2+n}+n} = \lim\limits_{n\to\infty}\frac{n}{n(\sqrt{1+\frac{1}{n}}+1)} = \frac{1}{2}$ ……………………………………..2p

$\lim\limits_{n\to\infty}(\sqrt[3]{n^3+n^2}-n) = \lim\limits_{n\to\infty}\frac{n^3+n^2-n^3}{\sqrt[3]{(n^3+n^2)^2}+n\sqrt[3]{n^3+n^2}+n^2} = \lim\limits_{n\to\infty}\frac{n^2}{n^2(\sqrt[3]{\left(1+\frac{1}{n}\right)^2}++\sqrt[3]{1+\frac{1}{n}}+1)} = \frac{1}{3}$ ……....2p

Atunci , limita din enunt este egala cu $\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$ ……………………….……………………..1p

SUBIECTUL III (7 puncte)

Se consideră matricea $A=\frac{1}{2\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}$. **Să se determine** $A^n=\begin{pmatrix}x_n & y_n\\ -y_n & x_n\end{pmatrix}$, $n\in\mathbb{N}^*$ **și să se calculeze limitele șirurilor** $(x_n)_n$ **și** $(y_n)_n$.

$A^n = \frac{1}{2}\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\cos\frac{\pi}{4} & \sin\frac{\pi}{4}\\ -\sin\frac{\pi}{4} & \cos\frac{\pi}{4}\end{pmatrix}$ …………………….…………………………………2p

Calculand si demonstrand prin inductie rezultatul se obtine ca:

$A^n = \frac{1}{2^n}\begin{pmatrix}\cos\frac{n\pi}{4} & \sin\frac{n\pi}{4}\\ -\sin\frac{n\pi}{4} & \cos\frac{n\pi}{4}\end{pmatrix}$ ……………………………………..…………….……..……3p

Se obtine ca: $x_n=\frac{1}{2^n}\cos\frac{n\pi}{4}$ si $y_n=\frac{1}{2^n}\sin\frac{n\pi}{4}$. Sirurile converg la 0, fiind produsul unui sir convergent la 0 cu un sir marginit. ……………………………………………...….………..2p

SUBIECTUL IV (7 puncte)

Se consideră șirul $(x_n)_{n\geq 0}$ definit prin $x_{n+1} = x_n + \frac{1}{e^{x_n}}$ și cu $x_0 \in \mathbb{R}$. Arătați că $\lim\limits_{n\to\infty} \frac{\ln n}{x_n} = 1$.

Monotonia $x_{n+1} - x_n = e^{-x_n} > 0 \Rightarrow x_n$ -strict crescator ……………………………………………1p
Marginirea: Daca x_n este marginit superior, atunci din Weierstrass, obtinem x_n convergent, si trecand la limita in relatia de recurenta, obtinem:

$l = l + e^{-l}$, cu $l \in \mathbb{R} \Rightarrow \lim\limits_{n\to\infty} x_n = +\infty$ ………………………………..……………………..2p

Aplicam, acum Cesaro-Stolz:

$\lim\limits_{n\to\infty} \frac{\ln n}{x_n} = \lim\limits_{n\to\infty} \frac{\ln(n+1) - \ln n}{x_n + 1 - x_n} = \lim\limits_{n\to\infty} ln \frac{n+1}{n} \cdot e^{x_n} = \lim\limits_{n\to\infty} \frac{e^{x_n}}{n} \cdot \ln\left(1 + \frac{1}{n}\right)^n = \lim\limits_{n\to\infty} \frac{e^{x_n}}{n}$ ……..……….2p

Aplic iarasi lema Cesaro-Stolz:

$\lim\limits_{n\to\infty} \frac{e^{x_n}}{n} = \lim\limits_{n\to\infty} \frac{e^{x_{n+1}} - e^{x_n}}{n+1-n} = \lim\limits_{n\to\infty} e^{x_n}\left(e^{x_{n+1} - x_n} - 1\right) = \lim\limits_{n\to\infty} \frac{1}{x_{n+1} - X_n}\left(e^{x_{n+1} - x_n} - 1\right) = 1$ …..…2p